EQUITY
LEARNING PLACE

Sec 4 Differentiation Revision II

1) If $y = (1+x)e^{3x}$, find the value of the constant k for which [6]

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + ky = 0$$

$$y = (1+x)e^{3x}$$

$$\frac{dy}{dx} = e^{3x} + (1+x)(3e^{3x})$$

$$\frac{dy}{dx} = e^{3x}(1+3+3x)$$

$$= e^{3x}(4+3x)$$

$$\frac{d^2y}{dx^2} = 3e^{3x}(4+3x) + e^{3x}(3)$$

$$= 3e^{3x}(4+3x+1)$$

$$= 3e^{3x}(5+3x)$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + ky = 0$$

$$3e^{3x}(5+3x) - 6e^{3x}(4+3x) + ke^{3x}(1+x) = 0$$

$$15+9x-24-18x+k+kx=0$$

$$kx-9x-9+k=0$$

$$\therefore k=9$$

Sec 4 Differentiation Revision II

2) Liquid is poured into a container at a rate of k m^3/s. The volume of liquid in the container is V m^3 where $V = \frac{1}{3}\pi h^2(3k - h)$ and h m is the depth of the liquid in the container. Find, in terms of k, the rate of increase of depth of the liquid when the depth of the liquid is $\frac{2k}{5}$ m. [4]

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$V = \frac{1}{3}\pi h^2(3k-h)$$

$$V = \pi h^2 k - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = 2\pi hk - \pi h^2$$

$$\frac{dh}{dt} = \frac{1}{2\pi hk - \pi h^2} \times k$$

when $h = \frac{2k}{5}$

$$\frac{dh}{dt} = \frac{1}{2\pi\left(\frac{2k}{5}\right)(k) - \pi\left(\frac{2k}{5}\right)^2}$$

$$= \frac{1}{\frac{4}{5}\pi k^2 - \frac{4}{25}\pi k^2}$$

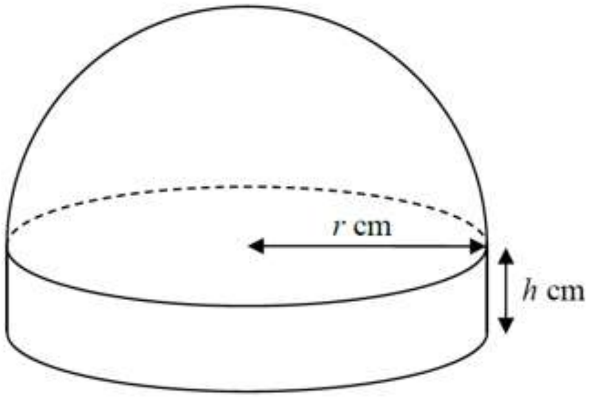
$$= \frac{25}{20\pi k^2 - 4\pi k^2}$$

$$= \frac{25}{16\pi k^2}$$

$\therefore$ rate of increase of depth is $\frac{25}{16\pi k^2}$ m/s

Sec 4 Differentiation Revision II

3) In the diagram, a solid model is made up of a hemisphere of radius r cm and a cylinder. The cylinder has a radius of r cm and a height of h cm.



i) Given that the volume of the model is 650 cm^3, express h in terms of r. [2]

ii) Given that the total surface area of the model is A cm^2, show that [2]

$$A = \frac{1300}{r} + \frac{5\pi r^2}{3}$$

iii) Given that r and h can vary, find the value of r for which A has a stationary value and determine whether this value of A is a maximum or a minimum. [5]

(i) $V = \frac{2}{3}\pi r^3 + \pi r^2 h$

$650 = \frac{2}{3}\pi r^3 + \pi r^2 h$

$650 - \frac{2}{3}\pi r^3 = \pi r^2 h$

$\frac{650}{\pi r^2} - \frac{2}{3}r = h$

(ii) $A = 2\pi r^2 + 2\pi r h + \pi r^2$

$A = 3\pi r^2 + 2\pi r\left(\frac{650}{\pi r^2} - \frac{2}{3}r\right)$

$A = 3\pi r^2 + \frac{1300}{r} - \frac{4}{3}\pi r^2$

$A = \frac{1300}{r} + \frac{5\pi r^2}{3}$

(iii) $A = \frac{1300}{r} + \frac{5\pi r^2}{3}$

$\frac{dA}{dr} = -\frac{1300}{r^2} + \frac{10}{3}\pi r$

when $\frac{dA}{dr} = 0$,

$0 = -\frac{1300}{r^2} + \frac{10}{3}\pi r$

$\frac{1300}{r^2} = \frac{10}{3}\pi r$

$3900 = 10\pi r^3$

$r^3 = \frac{390}{\pi}$

$r = \sqrt[3]{\frac{390}{\pi}}$

$= 4.988$

$\simeq 4.99 cm$

$\frac{d^2A}{dr^2} = \frac{2600}{r^3} + \frac{10}{3}\pi$

when $r = 4.988$,

$\frac{d^2A}{dr^2} > 0$

$\therefore$ A is a minimum when $r = 4.99$ cm

Sec 4 Differentiation Revision II

4) A curve has the equation

$$y = \frac{\sin x}{2 - \cos x}, \qquad 0 < x < \pi$$

a) Find an expression for $\frac{dy}{dx}$ in its simplest form. [2]

b) Find the integer value of x such that y is an increasing function. [4]

a) $y = \frac{\sin x}{2 - \cos x}$

$\frac{dy}{dx} = \frac{\cos x\,(2 - \cos x) - \sin x\,(\sin x)}{(2 - \cos x)^2}$

$= \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2}$

$= \frac{2\cos x - 1}{(2 - \cos x)^2}$

b) For y to be increasing,

$\frac{2\cos x - 1}{(2 - \cos x)^2} \geq 0$

$2\cos x - 1 \geq 0$

$\cos x \geq \frac{1}{2}$

$\therefore 0 < x \leq \frac{\pi}{3}$

So integer value of x is 1.

Sec 4 Differentiation Revision II

5) A curve is such that $\frac{d^2y}{dx^2} = 2(1-2x)$. The equation of the normal to the curve at the point $(-1, 7)$ is $9y = x + 64$. Find the equation of the curve. [5]

$\frac{d^2y}{dx^2} = 2 - 4x$

$\frac{dy}{dx} = \int 2 - 4x \, dx$

$= 2x - 2x^2 + c$

Eqn. of normal

$9y = x + 64$

$y = \frac{1}{9}x + \frac{64}{9}$

Grad. of normal $= \frac{1}{9}$

Grad of tangent $= -9$

when $x = -1$, $\frac{dy}{dx} = -9$

$-9 = -2 - 2 + c$

$c = -5$

$\frac{dy}{dx} = 2x - 2x^2 - 5$

$y = \int 2x - 2x^2 - 5 \, dx$

$y = x^2 - \frac{2}{3}x^3 - 5x + c$

when $x = -1$, $y = 7$

$7 = (-1)^2 - \frac{2}{3}(-1)^3 - 5(-1) + c$

$c = \frac{1}{3}$

$\therefore y = x^2 - \frac{2}{3}x^3 - 5x + \frac{1}{3}$